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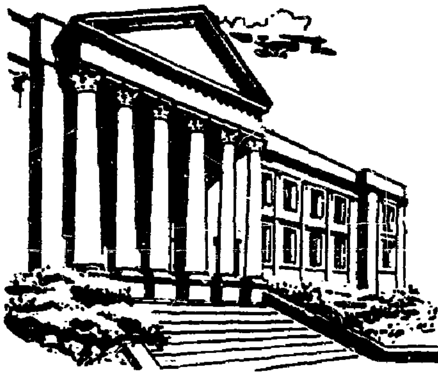

Interim Report

I-A2049-10

REFINEMENTS OF THE THEORY OF THE INFINITELY-LONG, SELF-ACTING, GAS-LUBRICATED JOURNAL BEARING

by
Harold G. Elrod, Jr.
Albert Burgdorfer

January 1960

Prepared under

Contract Nonr-2342(00)
Task NR 097-343

Jointly Supported by

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ATOMIC ENERGY COMMISSION
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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Presented at 1st International Symposium on Gas Lubricated Bearings at Washington, D.C.
October 26 to October 28, 1959

ABSTRACT

The lubrication equations for an arbitrary Newtonian fluid are derived directly from the general equations for conservation of mass, momentum, and energy. From the lubrication equations an inequality is obtained for the internal film temperature rise. The isothermal film equations are ~~then~~ derived, ~~Then~~ and, for perfectly-aligned self-acting journal bearings, a conservation equation is obtained. For gas bearings this condition gives:

$$\int_0^{2\pi} P^2 h^3 d\theta = \text{constant}$$

along the axis of the bearing. Application of this condition to the infinitely-long gas bearing gives more accurate pressure solutions for this case.

The Katto-Soda form of the differential equation for the infinitely-long bearing is solved by a series expansion in the eccentricity ratio, the first terms of which give the original, approximate Katto-Soda solution. In addition, solutions obtained numerically by digital computations are presented in graphical and tabular form for eccentricity ratios from $0 \to 0.9$ and compressible bearing parameter, λ_1 from $0 \to \infty$.

Design charts based on these calculations are provided.

TABLE OF CONTENTS

Nomenclature:

Symbol	Meaning
A	function of "ν" defined by eq. 68
A'	function of "ν" defined by eq. 71
B	function of "ν" defined by eq. 69
B'	function of "ν" defined by eq. 72
c	bearing clearance
C_p	specific heat of fluid
C_f	friction coefficient defined by eq. 80
C_L	load coefficient defined by eq. 78
D	diameter of shaft
h	film thickness; also, fluid enthalpy
k	thermal conductivity of fluid
L	length of very long bearing
M	Mach number corresponding to surface velocity of shaft; also, moment required to turn shaft
N_{Pr}	Prandtl number of fluid = $C_p\mu/k$
p	pressure
q	space-dependent heat addition per unit volume per unit time
R	radius of shaft
S	surface velocity of shaft
T	temperature
u	velocity of fluid in x-direction; also, $u = \psi - \tau$
v	velocity of fluid in y-direction; also, v_k stands for either $\sin(k\beta)$ or $\cos(k\beta)$.
W	load on bearing

Symbol	Meaning
x	coordinate in Cartesian system; also, $R\theta$
y	coordinate in Cartesian system
z	coordinate in Cartesian system
β	angle defined by eq. 54
β_o	angle corresponding to maximum film pressure
ϵ	eccentricity, defined by eq. 81
ζ	$\int_0^y \frac{dy}{\mu}$ at the point of maximum film pressure. See eq. 12.
θ	angle around bearing periphery, measured counter-clockwise from point of minimum film thickness.
κ	$(ph)_{max}$ See eq. 47.
μ	fluid viscosity
ν	a Katto-Soda bearing parameter, defined by eq. 57. also, kinematic viscosity of fluid.
ρ	mass density of fluid
τ	$1 - \epsilon\cos(\beta)$ See eq. 56.
ϕ	$\int_{\tau_w}^{\tau} k/\mu \, d\tau$ at the point of maximum pressure. See eq. 17.
χ	$\chi = \int \frac{\rho}{\mu} dp$ See eq. 37.
ψ	dimensionless pressure defined by eq. 53.
ω	angular velocity of shaft

Subscript or Superscript	Meaning
* (super.)	denotes standard reference quantity
o (sub.)	zeroth order, etc.
w (sub.)	wall

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REFINEMENTS OF THE THEORY OF THE INFINITELY-LONG, SELF-ACTING GAS-LUBRICATED JOURNAL BEARING

Harold G. Elrod, Jr.
Albert Burgdorfer

Introduction:

Gas-bearing technology has now a forty-five year history, dating back to the pioneer work of Harrison[1]. Moreover, in this field there has been a recent spurt of activity caused by the possibility of important applications. Nevertheless, in the course of progress towards useful devices, gaps in our knowledge have appeared, even in connection with the more elementary forms of lubricating equipment. The present paper endeavors to close a few such gaps which have appeared in connection with the infinitely-long, self-acting, gas-lubricated journal bearing. The theory of this type of bearing is reviewed, mathematical techniques for numerical development of the theory are discussed, and finally, improved design charts for engineering use are presented.

No attempt will be made here to review and assess the extensive contributions of prior workers on the subject of this paper. A recent comprehensive bibliography[2] of gas-lubrication research is available, to which the interested reader can refer. Two papers connected especially closely with the present research are those of Ausman[3] and of Katto and Soda[4]. In each of these papers, approximate solutions

of the gas-bearing equations are given, together with design charts based on these solutions. Reference to these works will be made subsequently when comparisons are made with the present results.

Basic Equations of Laminar Lubrication:

The general differential equations expressing the conservation of momentum, mass and energy for an arbitrary Newtonian fluid[5] constitute the foundation for the present analysis. Continuity of velocity and temperature is assumed at all fluid-solid interfaces. In a paper presented in this Symposium, the validity of these differential equations for a Newtonian fluid is examined by Reiner, and second-order corrections are proposed. The assumed boundary conditions are justifiable in most lubrication applications, but Burgdorfer[6] has shown that some gas bearings operate in the slip-flow regime. If the molecular mean free path does not exceed about five per cent of the radial clearance in the bearing, discontinuity effects at fluid-solid interfaces are not quantitatively very significant.

In Appendix I the basic equations of laminar lubrication are derived from the general conservation equations by means of a small-parameter technique in which the fluid film thickness-to-length ratio (h/L) serves as the small parameter. The resulting equations are as follows (See Nomenclature for meaning of symbols):

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y} \tag{1}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2}$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial y}\mu\frac{\partial w}{\partial y} \tag{3}$$

$$\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \mu\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \tag{4}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{5}$$

The fluid properties in these equations may be variable. Thus both liquids and gases are included.

Temperature Variation within Lubricating Film:

The strict solution of any lubrication problem would involve the coupling of the foregoing differential equations for the fluid with the differential equations of elasticity and of heat conduction appropriate for the shaft and bearing. To simplify the system of equations which must be considered, we shall here neglect both strain and temperature gradient within the shaft and bearing, and assume that shaft and bearing are at the same temperature. In cases where this latter assumption is justifiable, the internal temperature rise of the film is usually negligible, as will now be proved.

To investigate the magnitude of the transverse temperature rise, we consider the point of maximum film pressure. At this

point:
$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \tag{6}$$

In the absence of pressure gradient, eqs. 1, 3 and 4 simplify to:

$$\frac{\partial}{\partial y}\mu\frac{\partial u}{\partial y} = 0 \tag{7}$$

$$\frac{\partial}{\partial y}\mu\frac{\partial w}{\partial y} = 0 \tag{8}$$

$$\frac{\partial}{\partial y}k\frac{\partial T}{\partial y} = -\mu\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \tag{9}$$

Let us suppose that the surface at y = 0 has velocity components U, O, W in the x, y and z directions, respectively, and that the surface at y = h is stationary. The appropriate solutions of eqs. 7 and 8 are then readily found to be:

$$u = U\left(1 - \frac{\zeta}{\zeta_h}\right) \tag{10}$$

$$w = W\left(1 - \frac{\zeta}{\zeta_h}\right) \tag{11}$$

where:
$$\zeta \equiv \int_0^y \frac{dy}{\mu} \tag{12}$$

Substitution of eqs. 10 and 11 into eq. 9 gives:

$$\frac{\partial}{\partial y}k\frac{\partial T}{\partial y} = \frac{U}{\zeta_h}\frac{\partial u}{\partial y} + \frac{W}{\zeta_h}\frac{\partial w}{\partial y} \tag{13}$$

Integration yields:

$$k \frac{\partial T}{\partial y} = \frac{U}{\zeta_h} u + \frac{W}{\zeta_h} w + constant \qquad (14)$$

Using eqs. 10 and 11 again, we get:

$$\frac{k}{\mu} \frac{\partial T}{\partial \zeta} = \frac{S^2}{\zeta_h}\left(1 - \frac{\zeta}{\zeta_h}\right) + constant \qquad (15)$$

where: $$S^2 = U^2 + W^2 \qquad (16)$$

Now at the point of maximum pressure, k/μ can be regarded as a function of temperature only. Hence we define the new fluid property:

$$\phi = \int_{T_w}^{T} \frac{k}{\mu} dT \qquad (17)$$

Because both "k" and "μ" are inherently positive, "ϕ" increases monotonically with "T". With the temperatures at y = 0 and y = h equal, the net change in "ϕ" across the film is zero. Hence the integral of the right-hand side of eq. 15 must vanish. The constant is thus determined, and the final expression for the distribution of "ϕ" is:

$$\phi = \frac{S^2}{2} \frac{\zeta}{\zeta_h}\left(1 - \frac{\zeta}{\zeta_h}\right) \qquad (18)$$

The maximum value of "ϕ", which corresponds to the maximum value of temperature, occurs where $\zeta = \frac{1}{2}\zeta_h$

$$\therefore \phi_{max.} = \frac{S^2}{8} \qquad (19)$$

Equation 19 is perfectly general for the maximum value of "ϕ" at the point of maximum pressure in the film. We have not proved that this equation gives the maximum value of "ϕ" occurring anywhere in the bearing. However, it does give the value of $\phi_{max.}$ at a very important point, and this value must be at least representative of values occurring elsewhere in the film.

An examination of fluid properties discloses (with no observed exceptions) that k/μ increases with temperature for both liquids and gases. Hence k/μ is less in value at the film edges than internally. As a result, we can write the following useful inequality for checking transverse temperature variation in a lubricating film.

$$T_{max.} - T_w \leq \frac{S^2}{8(k/\mu)_0} \qquad (20)$$

In the case of a perfect gas with constant specific heats, the foregoing inequality can also be written as:

$$\frac{\Delta T}{T_w} \leq \frac{\gamma - 1}{2} N_{Pr} M^2 \cong \frac{M^2}{6} \qquad (21)$$

where "M" is the surface Mach number; i. e., the ratio of the surface velocity to the velocity of sound in the gas.

Isothermal Film Equations:

In many applications, eq. 20 (or 21) can be used to establish that the transverse temperature variation within the

lubricating film is negligible. In this case, the temperature everywhere within the film will be constant when the bearing and shaft are uniformly at the same temperature. Hence all fluid properties become functions of pressure only. Subject to this condition, the differential equation for the pressure in a lubricating film has been derived many places. We repeat the derivation here only for the sake of completeness.

In an isothermal film all fluid properties are independent of "y", since "p" is independent of "y". Thus eqs. 1 and 3 become:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{22}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial z} \tag{23}$$

The boundary conditions

$$\begin{aligned} u(0) &= U; \; u(h) = 0 \\ w(0) &= W; \; w(h) = 0 \end{aligned} \tag{24}$$

are satisfied by the following solutions of eqs. 22 and 23.

$$u = U\left(1 - \frac{y}{h}\right) + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - hy\right) \tag{25}$$

$$w = W\left(1 - \frac{y}{h}\right) + \frac{1}{2\mu}\frac{\partial p}{\partial z}\left(y^2 - hy\right) \tag{26}$$

The mass continuity equation (eq. 5) can be integrated with respect to "y" to give:

$$\int_0^h \frac{\partial}{\partial x}(\rho u)\, dy + \int_0^h \frac{\partial}{\partial z}(\rho w)\, dy = 0 \tag{27}$$

Equation 27 can alternatively be written as:

$$\frac{\partial}{\partial x}\rho\int_0^h u\,dy + \frac{\partial}{\partial z}\rho\int_0^h w\,dy = 0 \qquad (28)$$

because (a) the integration commutes with both derivatives

(b) the velocities vanish at the upper limit, y = h

(c) the fluid density is independent of "y".

We can now integrate the velocities given by eqs. 25 and 26, as required in eq. 28, to get:

$$\frac{\partial}{\partial x}\rho\left(\frac{Uh}{2} - \frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\rho\left(\frac{Wh}{2} - \frac{h^3}{12\mu}\frac{\partial p}{\partial z}\right) = 0 \qquad (29)$$

Or:

$$\frac{\partial}{\partial x}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial z} = 6\left(U\frac{\partial \rho h}{\partial x} + W\frac{\partial \rho h}{\partial z}\right) \qquad (30)$$

There is a useful exact analogy which should here be pointed out between the pressure in a lubricating film and the temperature in a constant density, moving fluid. To illustrate this fact, let us rearrange eq. 30 in the following form:

$$6h\left(\frac{dp}{d\rho}\right)_T\left(U\frac{\partial p}{\partial x} + W\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial x}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial z} + $$
$$-6\rho\left(U\frac{\partial h}{\partial x} + W\frac{\partial h}{\partial z}\right) \qquad (31)$$

In this equation, "p" plays the role of temperature, while "U" and "W" retain their meaning as velocities. The quantity $6h\left(\frac{dp}{d\rho}\right)_T$ is intrinsically positive, because the film thickness is certainly positive, and $\left(\frac{dp}{d\rho}\right)_T$, the "Newtonian velocity of

sound," is positive for any mechanically stable substance. Hence we let $6h\left(\frac{dp}{d\rho}\right)_T$ play the role of volumetric specific heat. The quantity $\frac{\rho h^3}{\mu}$ is inherently positive, so we let is serve in the analogy as thermal conductivity. Finally, the last group of terms, being the product of "ρ" with a space-dependent function $6\left(U\frac{\partial h}{\partial x}+W\frac{\partial h}{\partial z}\right)$, we take to be a temperature and space-dependent internal source of heat. Thus the equation in convective heat transfer analogous to eq. 31 is:

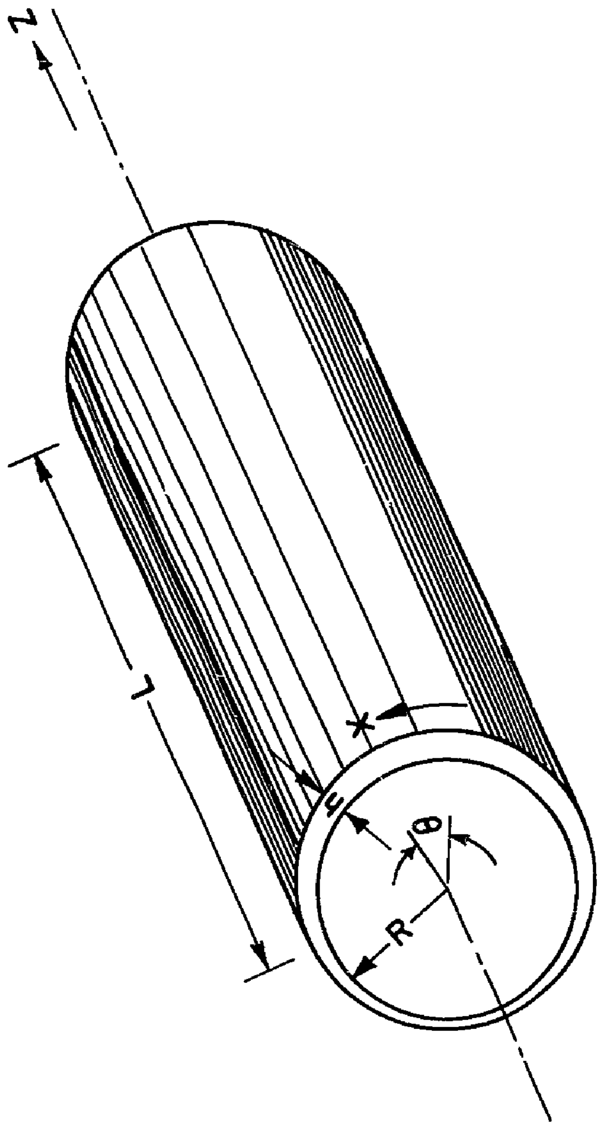
$$\rho c\left(U\frac{\partial T}{\partial x}+W\frac{\partial T}{\partial z}\right)=\frac{\partial}{\partial x}k\frac{\partial T}{\partial x}+\frac{\partial}{\partial z}k\frac{\partial T}{\partial z}+g(T,x,z) \tag{32}$$

The terms in this last equation have been defined by the preceding discussion. This analogy enables one to exploit intuition and solutions acquired in another, well-established field.

<u>Differential Equation and Mass Content of Infinitely-Long Self-Acting Journal Bearing</u>:

Figure 1 shows schematically a long, self-acting journal bearing. Here "x" denotes peripheral distance measured in a counter-clockwise direction, and "z" denotes axial distance. The use of "x" to denote a curvilinear, rather than rectilinear distance, has been examined by Elrod[7] for the case of fluids with constant properties, and the effect has been found to be exceedingly slight. There is no reason to suppose that otherwise will be the case for fluids with variable properties.

When a journal bearing is perfectly aligned, with both ends



SCHEMATIC DRAWING OF LONG, SELF-ACTING JOURNAL BEARING

exposed to the same ambient pressure, there is a simple conservation condition which can be derived from eq. 30. For this situation:

$$h = h(x) \text{ and } \quad W = 0 \tag{33}$$

and eq. 30 reduces to:

$$\frac{\partial}{\partial z}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial z} = 6\, U\frac{\partial \rho h}{\partial x} - \frac{\partial}{\partial x}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x} \tag{34}$$

Now all flow quantities are cyclic in "x"; i. e., their net change circumferentially around the shaft is zero. Hence, if we integrate eq. 34 around the shaft, the right-hand side reduces to zero, and we are left with:

$$\oint \frac{\partial}{\partial z}\frac{\rho h^3}{\mu}\frac{\partial p}{\partial z}\, dx = \frac{\partial}{\partial z}\oint \frac{\rho h^3}{\mu}\frac{\partial p}{\partial z}\, dx = 0 \tag{35}$$

Or:

$$\oint h^3 \frac{\rho}{\mu}\frac{\partial p}{\partial z}\, dx = A, \text{ a constant} \tag{36}$$

Since both "ρ" and "μ" are functions of "p" only, we can define a new function:

$$\chi \equiv \int \frac{\rho}{\mu}\, dp \tag{37}$$

in terms of which eq. 36 becomes:

$$\oint \frac{\partial}{\partial z} h^3(x)\chi\, dx = \frac{\partial}{\partial z}\oint h^3\chi\, dx = A \tag{38}$$

$$\therefore \quad \oint h^3 \chi\, dx = Az + B \tag{39}$$

The cyclic integral in eq. 39 must have the same value at z = 0 and z = L. Hence the constant "A" must be zero. The following conservation equation then results:

$$\oint h^3 \chi \, dx = \chi_a \oint h^3 \, dx \tag{40}$$

where χ_a corresponds to ambient pressure, and may be taken as zero, if desired. Since, very closely,

$$h = c\,(1 + \epsilon \cos \theta\,) \tag{41}$$

we have:

$$\oint h^3 dx = c^3 R \oint (1+\epsilon\cos\theta)^3 d\theta = 2\pi R c^3 (1+\tfrac{3}{2}\epsilon^2) \tag{42}$$

$$\therefore \oint h^3 \chi \, dx = 2\pi R c^3 (1+\tfrac{3}{2}\epsilon^2)\chi_a \tag{43}$$

This last equation applies to any Newtonian fluid, in the absence of cavitation.

If the lubricant is a perfect gas, then its viscosity is independent of pressure, and, at constant temperature, its density is proportional to the pressure. Hence we can take χ equal to p^2. That is;

$$\oint h^3 p^2 dx = \text{constant} = 2\pi R c^3 (1+\tfrac{3}{2}\epsilon^2)\, p_a^2 \tag{44}$$

Since this condition applies to a gas journal bearing, however long, it applies to the "infinitely long" bearing. That is, it applies to the central region of a long bearing, far from either end, where the pressure is essentially a function of "x" only. It is only in terms of this region that we may think of the "infinitely long bearing", since otherwise we can establish no connection with an ambient pressure, and hence no connection with any physically real problem.

Equation 43, with its various specializations, such as eq. 44, furnishes a condition which sets the general pressure level, and hence the mass content, of any continuous-film, infinite bearing. It is an important relation in the theory of the infinite gas bearing, and has not, so far as the authors are aware, been previously employed. In the case of non-cavitating films of incompressible fluid, the mass content of the bearing is, of course, known. Also, if the viscosity is constant, the load-carrying capacity of the bearing can be computed without knowledge of the general pressure level. However, eq. 43 still applies, and might be useful for predicting the <u>onset</u> of cavitation, or for refined treatments with variable viscosity.

The differential equation for the infinitely-long bearing is obtained from eq. 34 by eliminating z-derivatives. It is:

$$\frac{d}{dx}\frac{\rho h^3}{\mu}\frac{dp}{dx} = 6\,U\,\frac{d\rho h}{dx} \tag{45}$$

Integrating once, we obtain the following first-order non-linear differential equation whose solution involves specification of two constants, one of which is shown explicitly.

$$\frac{\rho h^3}{\mu}\frac{dp}{dx} = 6\,U\rho h + constant \tag{46}$$

The two constants are determined by requiring (a) that the solution be cyclic in "x" and (b) that it satisfy the condition given by eq. 43.

<u>Series Solution for the Pressure in Infinitely-Long, Self-Acting Gas-Lubricated Journal Bearing</u>

For gas-lubricated bearings, eq. 46 can be readily put in the form:

$$\frac{dp}{d\theta} = \frac{6\mu UR}{h^2}\left[1 - \frac{\kappa}{ph}\right] \tag{47}$$

where $\theta = x/R$ and "κ" is the value of the product "ph" at a stationary point for the pressure. When the pressure in the film varies and the film thickness is given by eq. 41, there are two such stationary points, one corresponding to the maximum pressure "p_1" and one corresponding to the minimum pressure "p_2". Since "κ" is limited to a single value, we must have:

$$p_1 h_1 = p_2 h_2 \tag{48}$$

Or:

$$\frac{p_1}{p_2} = \frac{h_1}{h_2} \tag{49}$$

Now the ratio of h_1/h_2 is certainly less than the maximum value of film thickness $c(1+\epsilon)$ divided by the minimum value of film thickness $c(1-\epsilon)$. Hence:

$$\frac{p_1}{p_2} \leq \frac{1+\epsilon}{1-\epsilon} \tag{50}$$

From eq. 44 it is easily seen that p_1 is greater than p_a and that p_2 is less than p_a. Then from eq. 50 we get:

$$p_1 - p_2 \leq \left\{\frac{1+\epsilon}{1-\epsilon} - 1\right\} p_2 = \frac{2\epsilon}{1-\epsilon}\, p_2 \tag{51}$$

$$\therefore\ p_1 - p_2 \leq \frac{2\epsilon}{1-\epsilon}\, p_a \tag{52}$$

It is evident from eq. 52 that the load-carrying capacity of

a gas-lubricated journal bearing has a limit which cannot be exceeded through changes in gas or in speed, but only through changes in allowable eccentricity or in ambient pressure.

Closed-form solutions of eq. 47 do not exist, but approximate analytical solutions can be very helpful. Both Ausman[3] and Katto and Soda[4] have developed such solutions. In the analysis which follows, a series expansion for the film pressure will be obtained, the first terms of which give the Katto and Soda solution, with succeeding terms giving refinements thereof. Let us begin by introducing the Katto and Soda variables. They are:

$$\psi \equiv \frac{pc}{K}(1-\epsilon^2) \tag{53}$$

$$\cos\beta \equiv \frac{\epsilon + \cos\theta}{1+\epsilon\cos\theta} \tag{54}$$

In terms of these new variables, the differential equation corresponding to eq. 47 is:

$$\frac{d\psi}{d\beta} = \frac{1}{\nu}\,\tau\left[1-\frac{\tau}{\psi}\right] \tag{55}$$

where:
$$\tau = 1-\epsilon\cos\beta \tag{56}$$

and:
$$\nu = \frac{Kc\sqrt{1-\epsilon^2}}{6\mu UR} \tag{57}$$

Examining eq. 55, we note that when $\epsilon = 0$, the only cyclic solution is $\psi = 1 = \tau$. On the other hand, when the parameter $\nu \rightarrow 0$, we know the pressure variation must nevertheless be constrained by relations 50-52. Then if the pressure derivatives

are also to be bounded, we observe from eq. 55 that "ψ" must again approach "τ". In view of these facts, we hypothesize $u = \psi - \tau$ as a small quantity and attempt an expansion in the form:

$$u = \epsilon u_1(\beta) + \epsilon^2 u_2(\beta) + \epsilon^3 u_3(\beta) + \cdots \qquad (58)$$

In terms of "u" the differential equation becomes:

$$\frac{du^2}{d\beta} + 2\tau\frac{du}{d\beta} + 2u\frac{d\tau}{d\beta} + 2\tau\frac{d\tau}{d\beta} = \frac{2}{\nu}\tau u \qquad (59)$$

Now in terms of the series expansion in eq. 58,

$$u^2 = \sum_{i,j} \epsilon^{i+j} u_i u_j = \sum_{n=0}^{\infty} \epsilon^n \sum_{j=0}^{\infty} u_j u_{n-j} \qquad (60)$$

Here the convention is adopted that $u_n = 0$ for $n \leq 0$.

Using the expansions from eqs. 58 and 60 in eq. 59, we get:

$$\sum_{n=0}^{\infty}\epsilon^n \frac{d}{d\beta}\sum_{j=0}^{\infty} u_j u_{n-j} + 2\sum_{n=0}^{\infty}\epsilon^n\frac{du_n}{d\beta} - 2\cos\beta\sum_{n=0}^{\infty}\epsilon^n\frac{du_{n-1}}{d\beta} + \qquad (61)$$

$$+ 2\sin\beta\sum_{n=0}^{\infty}\epsilon^n u_{n-1} + 2\sum_{n=0}^{\infty}\epsilon^n\chi_n(\beta) = \frac{2}{\nu}\sum_{n=0}^{\infty}\epsilon^n u_n - \frac{2}{\nu}\cos\beta\sum_{n=0}^{\infty}\epsilon^n u_{n-1}$$

where: $\chi_0(\beta) = 0$; $\chi_1(\beta) = \sin\beta$; $\chi_2(\beta) = -\frac{1}{2}\sin 2\beta$

$$\chi_n(\beta) = 0, \quad n \geq 3 \qquad (62)$$

When the coefficients of the various powers of "ϵ" are equated to zero in eq. 61, the following sequence of linear differential equations results.

$$\frac{d}{d\beta}(u_n - \cos\beta\, u_{n-1}) - \frac{1}{\nu}(u_n - \cos\beta\, u_{n-1}) = -\chi_n(\beta) - \frac{1}{2}\sum_{j=0}^{\infty} u_j u_{n-j} \qquad (63)$$

The cyclic solution for $(u_n - u_{n-1}\cos\beta)$ is given by:

$$u_n - u_{n-1}\cos\beta = -e^{\frac{\beta}{\nu}}\int^{\beta} e^{-\frac{\beta}{\nu}}\left\{\chi_n + \frac{1}{2}\frac{d}{d\beta}\sum_{j=0}^{\infty} u_j\, u_{n-j}\right\}d\beta \quad (64)$$

where the constant of integration is to be taken as zero to avoid an exponentially-increasing, non-periodic solution.

The products $u_n u_{n-j}$ can all be reduced trigonometrically to a sum of terms of the form $\cos(k\beta)$ or $\sin(k\beta)$, where "k" is an integer. With $v_k(\beta)$ standing for either of these functions, the following indefinite integral formula is easily derived:

$$-e^{\frac{\beta}{\nu}}\int e^{-\frac{\beta}{\nu}} v_k(\beta)\, d\beta = \frac{\nu}{1+k^2\nu^2}\left[v_k(\beta) + \nu\frac{dv_k}{d\beta}\right] \quad (65)$$

With the aid of this formula we find from eq. 64:

$$u_1 = \frac{\nu}{1+\nu^2}\left[\sin\beta + \nu\cos\beta\right] \quad (65)$$

$$u_2 = u_1\cos\beta - e^{\frac{\beta}{\nu}}\int^{\beta} e^{-\frac{\beta}{\nu}}\left\{-\frac{1}{2}\sin 2\beta + \frac{1}{2}\frac{du_1^2}{d\beta}\right\}d\beta \quad (66)$$

By squaring eq. 65 we find the terms in the curly brackets of the integral in eq. 66 to be:

$$-\frac{1}{2}\sin 2\beta + \frac{1}{2}\frac{du_1^2}{d\beta} = A\sin 2\beta + B\cos 2\beta \quad (67)$$

where:

$$A = -\frac{1}{2} + \frac{1}{2}\nu^2\frac{(1-\nu^2)}{(1+\nu^2)^2} \quad (68)$$

$$B = \frac{\nu^3}{(1+\nu^2)^2} \quad (69)$$

Applying eq. 66 again, we get:

$$u_2 = u_1 \cos\beta + \frac{\nu}{1+4\nu^2}\left[(A-2\nu B)\sin 2\beta + (B+2\nu A)\cos 2\beta\right] \quad (70)$$

$$A - 2\nu B \equiv A' = -\frac{1}{2}\,\frac{(1+\nu^2+6\nu^4)}{(1+\nu^2)^2} \quad (71)$$

$$B + 2\nu A \equiv B' = -\nu\left[\frac{1+2\nu^4}{(1+\nu^2)^2}\right] \quad (72)$$

Substituting the expressions for u_1 and u_2 in series 58, we obtain the following expression for "ψ", valid to $O(\epsilon^2)$.

$$\psi = 1 - \epsilon\cos\beta + \epsilon(1+\epsilon\cos\beta)\frac{\nu}{1+\nu^2}(\sin\beta + \nu\cos\beta) + \frac{\epsilon^2\nu}{1+4\nu^2}\left(A'\sin 2\beta + B'\cos 2\beta\right) + \ldots\ldots \quad (73)$$

To $O(\epsilon)$ eq. 73 gives precisely the Katto-Soda solution; i. e.,

$$\psi = 1 - \frac{\epsilon}{1+\nu^2}\cos\beta + \frac{\epsilon\nu}{1+\nu^2}\sin\beta \quad (74)$$

Typical circumferential profiles of the dimensionless pressure "ψ" are shown in Figs. 2 and 3, as computed from eq. 73. These profiles are compared with others obtained by an accurate numerical method, presently to be described. It will be observed that the series solution is most accurate for small "ν" and small "ϵ". This characteristic was anticipated in the formulation of series 58. It will also be observed that the series solution is scarcely of adequate accuracy for the operating condition defined by $\nu = 1$, $\epsilon = 0.8$.

Digital Computer Solutions:

In view of the difficulty in finding approximate solutions of the one-dimensional gas bearing equations with sufficient accuracy, it was decided to attack the problem directly by using standard numerical techniques to solve eq. 55. The method employed was the interative scheme of Clippinger and Dimsdale[8]. The angle "β_o" corresponding to a maximum value of ψ was estimated from the Katto and Soda solution. Then the solution was started at this angle and continued in a clockwise direction with an angular increment of $\pi/30$. Three "rotations" of the computer usually sufficed to produce a solution periodic to within six significant figures. Dimensionless pressure profiles were obtained for five different eccentricities (ϵ = 0.2, 0.4, 0.6, 0.8, 0.9) with each of four different values of the speed parameter "ν" (ν = 1.00, 0.5, 0.25, 0.125). Values of "ψ" are listed in Tables 2-5. They are believed to be correct to within one digit in the last figure tabulated.

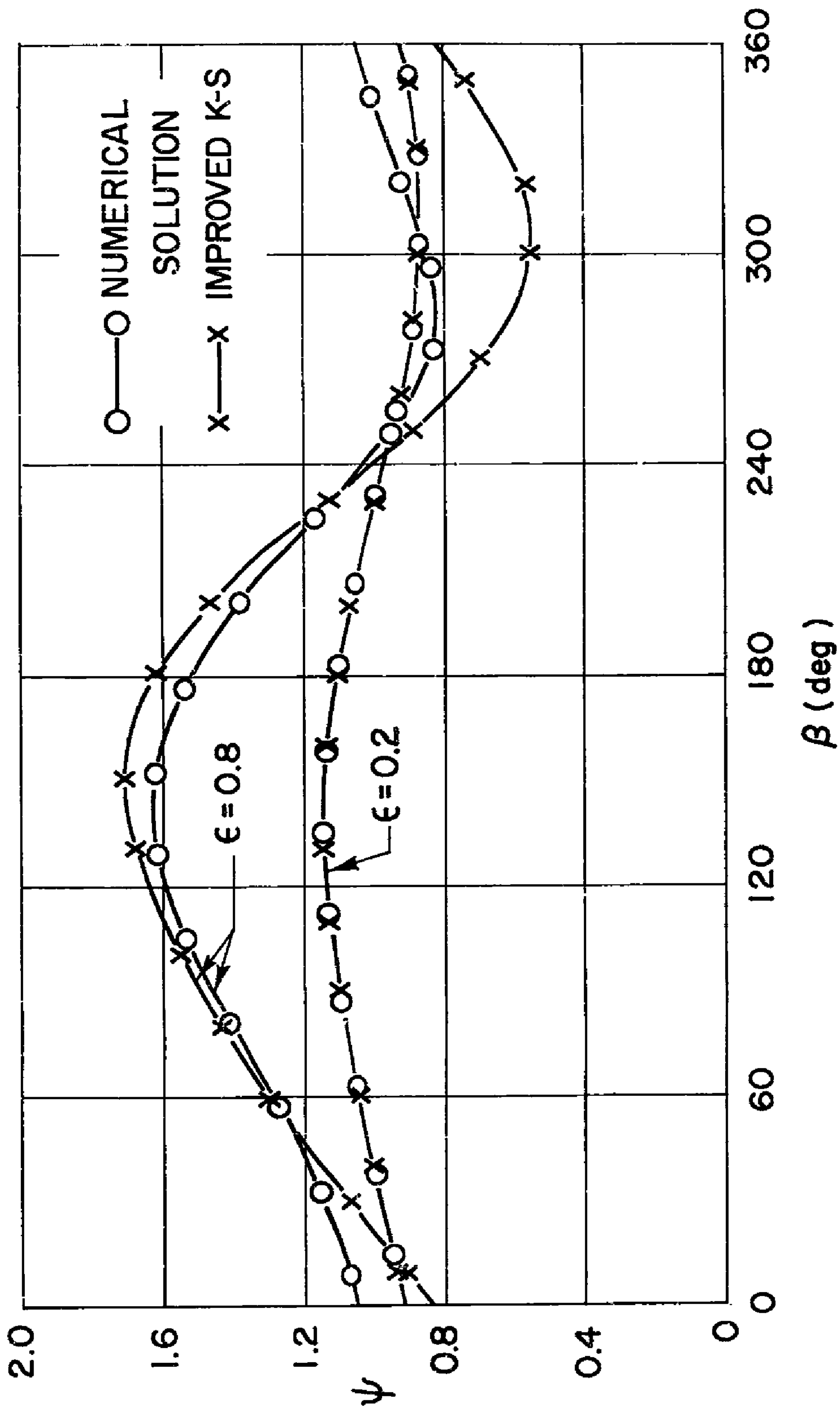
The accuracy of both the series and digital computer computations is confirmed by the close agreement between the results of both for small "ν" and "ϵ", as shown in Figs. 2 and 3. Actually, the derivation of the series was purely formal, no proof of convergence being offered. As a check on convergence, the following case was computed by both methods:

$$\nu = 1.0 \; ; \quad \epsilon = 0.05$$

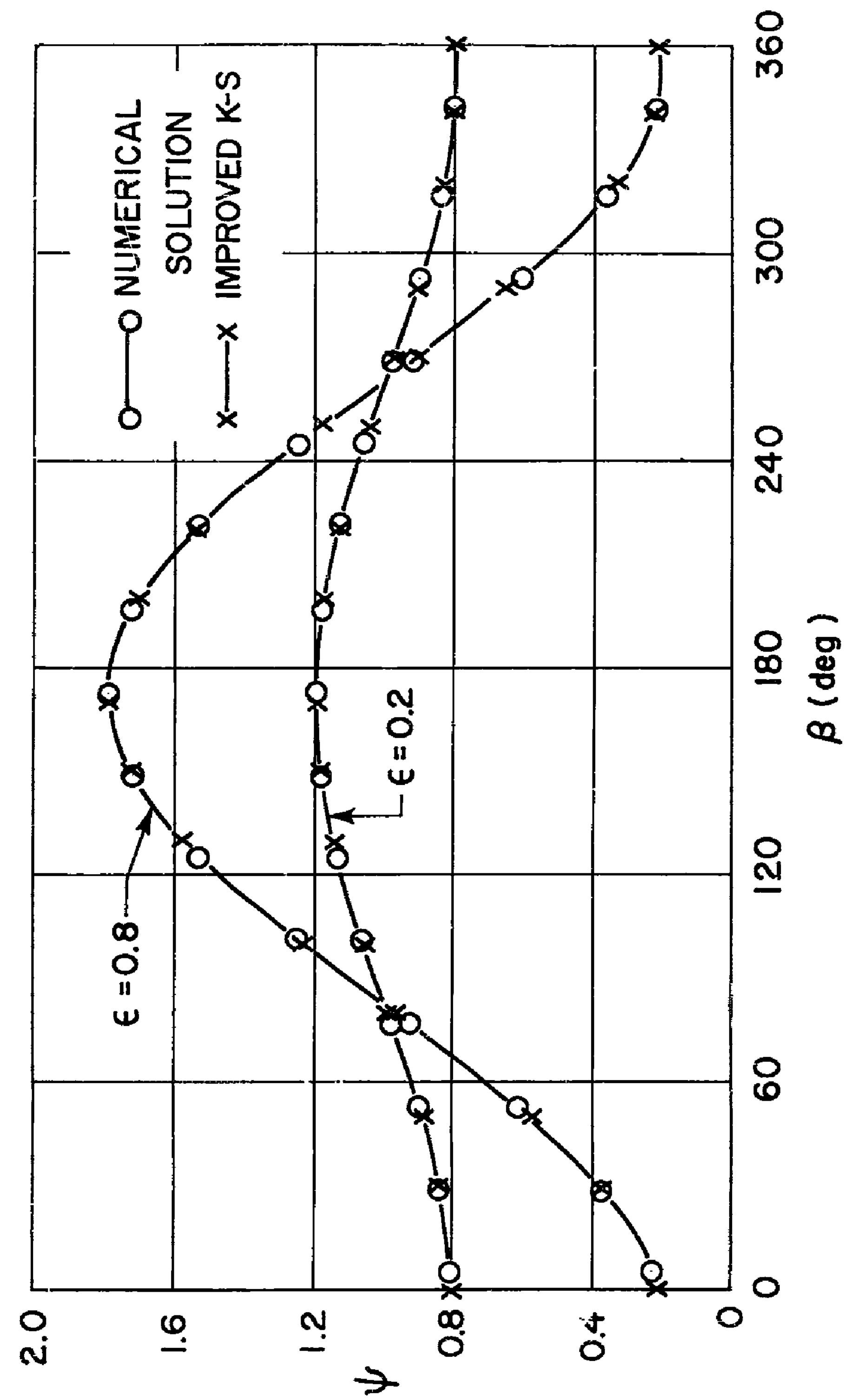
The results are compared below in the neighborhood of the pressure extremes. It will be observed that the inclusion of terms beyond the original Katto-Soda solution does yield a significant improvement in precision, tending to confirm the assumption of convergence of the series.

TABLE 1 : COMPARISON OF DIGITAL COMPUTER AND SERIES SOLUTIONS FOR THE CASE $\nu = 1.0$; $\epsilon = 0.05$

	Values of "ψ"		
β (rad)	Eq. 74	Eq. 73	Computer
2.40855	1.03531	1.03583	1.03580
2.35619	1.03536	1.03586	1.03583
2.30383	1.03531	1.03578	1.03575
5.55015	0.96469	0.96522	0.96525
5.49779	0.96465	0.96514	0.96518
5.44543	0.96469	0.96517	0.96520



COMPARISON OF IMPROVED K-S SOLUTION WITH EXACT NUMERICAL SOLUTION $\nu = 1.00$



COMPARISON OF IMPROVED K-S SOLUTION WITH EXACT NUMERICAL SOLUTION $\nu = 0.125$

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TABLE 2

PERIODIC SOLUTIONS OF THE KATTO-SODA DIFFERENTIAL EQUATION

$$\frac{d\psi}{d\beta} = \frac{\tau}{\nu}\left[1 - \frac{\tau}{\psi}\right]; \quad \tau \equiv 1 - \epsilon\cos\beta$$

$$\nu = 1.00$$

$\epsilon \rightarrow$	0.20	0.40	0.60	0.80	0.90
$\cos\beta_o \rightarrow$	-0.707	-0.629	-0.629	-0.629	-0.629
$\frac{15}{\pi}(\beta_o-\beta)$	ψ	ψ	ψ	ψ	ψ
0	1.148	1.300	1.459	1.621	1.703
1	1.144	1.285	1.435	1.589	1.667
2	1.134	1.261	1.399	1.541	1.613
3	1.119	1.228	1.352	1.481	1.547
4	1.100	1.189	1.298	1.413	1.473
5	1.078	1.146	1.239	1.343	1.399
6	1.053	1.099	1.178	1.274	1.329
7	1.026	1.052	1.118	1.212	1.269
8	0.999	1.004	1.061	1.158	1.221
9	0.972	0.958	1.008	1.113	1.186
10	0.946	0.914	0.957	1.075	1.162
11	0.928	0.873	0.910	1.041	1.142
12	0.902	0.836	0.863	1.006	1.121
13	0.886	0.805	0.820	0.966	1.092
14	0.874	0.783	0.780	0.922	1.053
15	0.869	0.771	0.751	0.875	1.004
16	0.870	0.774	0.738	0.836	0.953
17	0.878	0.791	0.748	0.816	0.915
18	0.892	0.823	0.784	0.829	0.906
19	0.912	0.868	0.842	0.878	0.937
20	0.937	0.922	0.918	0.959	1.008
21	0.966	0.982	1.004	1.059	1.106
22	0.996	1.043	1.094	1.169	1.220
23	1.027	1.104	1.183	1.279	1.337
24	1.056	1.160	1.126	1.381	1.447
25	1.083	1.209	1.335	1.471	1.544
26	1.107	1.249	1.392	1.543	1.622
27	1.125	1.279	1.434	1.595	1.679
28	1.139	1.298	1.459	1.625	1.711
29	1.146	1.305	1.467	1.634	1.719

Note: β_o is in the second quadrant.

TABLE 3

PERIODIC SOLUTIONS OF THE KATTO-SODA DIFFERENTIAL EQUATION

$$\frac{d\psi}{d\beta} = \frac{\tau}{\nu}\left[1 - \frac{\tau}{\psi}\right]; \quad \tau \equiv 1 - \epsilon \cos\beta$$

$$\nu = 0.50$$

$\epsilon \rightarrow$	0.20	0.40	0.60	0.80	0.90
$\cos\beta_0 \rightarrow$	-0.843	-0.843	-0.843	-0.843	-0.843
$\frac{15}{\pi}(\beta_0-\beta)$	ψ	ψ	ψ	ψ	ψ
0	1.179	1.360	1.542	1.724	1.815
1	1.171	1.343	1.317	1.690	1.777
2	1.156	1.314	1.472	1.631	1.711
3	1.135	1.272	1.410	1.551	1.621
4	1.108	1.220	1.335	1.453	1.512
5	1.076	1.160	1.239	1.343	1.391
6	1.042	1.094	1.156	1.227	1.265
7	1.006	1.026	1.061	1.111	1.142
8	0.970	0.958	0.967	1.002	1.029
9	0.935	0.891	0.877	0.903	0.933
10	0.902	0.828	0.793	0.818	0.858
11	0.874	0.772	0.716	0.745	0.803
12	0.851	0.724	0.648	0.683	0.762
13	0.835	0.687	0.589	0.624	0.725
14	0.825	0.664	0.544	0.566	0.681
15	0.824	0.657	0.521	0.511	0.624
16	0.831	0.669	0.526	0.475	0.563
17	0.847	0.698	0.563	0.484	0.528
18	0.869	0.744	0.629	0.547	0.553
19	0.898	0.803	0.718	0.652	0.643
20	0.931	0.871	0.821	0.784	0.777
21	0.968	0.945	0.932	0.928	0.933
22	1.006	1.021	1.046	1.077	1.096
23	1.043	1.096	1.157	1.223	1.258
24	1.079	1.167	1.460	1.359	1.409
25	1.111	1.229	1.353	1.479	1.544
26	1.138	1.282	1.430	1.580	1.656
27	1.159	1.323	1.489	1.657	1.742
28	1.173	1.350	1.528	1.708	1.798
29	1.180	1.362	1.546	1.730	1.823

Note: β_0 is in the second quadrant.

TABLE 4

PERIODIC SOLUTIONS OF THE KATTO-SODA DIFFERENTIAL EQUATION

$$\frac{d\psi}{d\beta} = \frac{\tau}{\nu}\left[1 - \frac{\tau}{\psi}\right]; \quad \tau \equiv 1 - \epsilon \cos\beta$$

$\nu = 0.25$

$\epsilon \rightarrow$	0.20	0.40	0.60	0.80	0.90
$\cos\beta_0 \rightarrow$	-0.940	-0.940	-0.940	-0.940	-0.956
$\frac{15}{\pi}(\beta_0-\beta)$	ψ	ψ	ψ	ψ	ψ
0	1.193	1.386	1.580	1.773	1.874
1	1.185	1.369	1.554	1.740	1.845
2	1.168	1.337	1.506	1.675	1.781
3	1.145	1.290	1.437	1.583	1.684
4	1.115	1.232	1.349	1.468	1.562
5	1.080	1.163	1.248	1.335	1.417
6	1.042	1.088	1.137	1.190	1.258
7	1.002	1.009	1.021	1.039	1.092
8	0.962	0.930	0.905	0.889	0.926
9	0.924	0.854	0.793	0.747	0.768
10	0.888	0.783	0.690	0.618	0.628
11	0.858	0.721	0.598	0.506	0.509
12	0.833	0.672	0.522	0.412	0.417
13	0.816	0.636	0.465	0.335	0.347
14	0.808	0.617	0.432	0.279	0.289
15	0.807	0.615	0.426	0.253	0.235
16	0.816	0.632	0.450	0.275	0.209
17	0.832	0.666	0.501	0.342	0.250
18	0.856	0.715	0.576	0.443	0.350
19	0.887	0.777	0.670	0.569	0.484
20	0.923	0.849	0.779	0.713	0.640
21	0.962	0.927	0.896	0.868	0.811
22	1.002	1.008	1.017	1.029	0.990
23	1.042	1.088	1.137	1.188	1.170
24	1.081	1.165	1.251	1.338	1.342
25	1.116	1.234	1.354	1.475	1.499
26	1.145	1.293	1.441	1.591	1.635
27	1.169	1.339	1.510	1.682	1.745
28	1.185	1.371	1.557	1.744	1.823
29	1.193	1.387	1.581	1.775	1.867

Note: β_0 is in the second quadrant.

TABLE 5

PERIODIC SOLUTIONS OF THE KATTO-SODA DIFFERENTIAL EQUATION

$$\frac{d\psi}{d\beta} = \frac{\tau}{\nu}\left[1 - \frac{\tau}{\psi}\right]; \quad \tau \equiv 1 - \epsilon \cos\beta$$

$$\nu = 0.125$$

$\epsilon \rightarrow$	0.20	0.40	0.60	0.80	0.90
$\cos\beta_0 \rightarrow$	-0.992	-0.992	-0.992	-0.992	-0.992
$\frac{15}{\pi}(\beta_0-\beta)$	ψ	ψ	ψ	ψ	ψ
0	1.198	1.397	1.595	1.794	1.893
1	1.194	1.388	1.582	1.777	1.874
2	1.181	1.363	1.544	1.726	1.817
3	1.161	1.322	1.483	1.644	1.724
4	1.133	1.266	1.400	1.534	1.601
5	1.100	1.200	1.300	1.402	1.452
6	1.062	1.125	1.188	1.252	1.284
7	1.021	1.044	1.067	1.092	1.105
8	0.980	0.961	0.944	0.929	0.922
9	0.939	0.880	0.823	0.769	0.743
10	0.901	0.804	0.710	0.619	0.576
11	0.868	0.737	0.608	0.486	0.428
12	0.840	0.681	0.524	0.374	0.306
13	0.819	0.639	0.460	0.287	0.211
14	0.806	0.612	0.419	0.230	0.146
15	0.802	0.603	0.405	0.208	0.114
16	0.806	0.612	0.418	0.225	0.129
17	0.819	0.638	0.457	0.278	0.190
18	0.840	0.680	0.521	0.364	0.287
19	0.868	0.736	0.605	0.477	0.414
20	0.901	0.803	0.707	0.612	0.566
21	0.939	0.880	0.821	0.764	0.736
22	0.980	0.961	0.943	0.926	0.918
23	1.021	1.044	1.067	1.091	1.104
24	1.062	1.125	1.188	1.252	1.284
25	1.100	1.200	1.301	1.402	1.453
26	1.133	1.267	1.401	1.535	1.602
27	1.161	1.322	1.483	1.645	1.726
28	1.181	1.363	1.545	1.727	1.818
29	1.194	1.388	1.583	1.777	1.874

Note: β_0 is in the second quadrant.

In order that the computed data on pressure distributions may be useful, it is necessary to relate the internal pressures of the infinite bearing to some ambient pressure by means of the mass content rule. When a change is made from conventional variables to those employed by Katto and Soda, the mass content rule (eq. 44) gives:

$$\oint p^2 h^3 d\theta = \kappa^2 c (1-\epsilon^2)^{3/2} \oint \frac{\psi^2}{r^4} d\beta \tag{75}$$

$$= 2\pi c^3 (1 + \tfrac{3}{2}\epsilon^2) p_a^2 \tag{76}$$

Or:

$$\frac{\kappa}{p_a c} = \sqrt{\frac{2\pi(1+3/2\,\epsilon^2)}{(1-\epsilon^2)^{3/2}\oint \frac{\psi^2}{r^4} d\beta}} \tag{77}$$

For the data tabulated in Tables 2-5 the required integration in eq. 77 was performed using Simpson's rule with an angular interval of $\pi/15$.

With $\kappa/p_a c$ known, the pressure ratio p/p_a can readily be found in terms of operating variables. Thus:

$$p/p_a = \frac{\kappa}{p_a c} \frac{\psi}{1-\epsilon^2} \tag{78}$$

$$\nu = \frac{\kappa}{p_a c} \frac{p_a c^2 \sqrt{1-\epsilon^2}}{6\mu U R} \tag{79}$$

To prepare Design Charts in terms of conventional bearing parameters, other integrations of the ψ-curves are required. For these integrations Simpson's rule was again used with an angular interval of $\pi/15$.

Design Charts:

The steady-state performance of an infinitely-long, self-acting, gas-lubricated journal bearing can be completely expressed in terms of certain well-known bearing parameters. In order that the subject matter of this section may be self-contained, these parameters will here be defined in terms of physical quantities which are easily interpreted. These parameters can be readily found from the ψ -curves, with the use of eqs. 77-79.

Load Parameter, C_L

This parameter represents the ratio of the load, W, which is actually supported by the bearing to the force which would be exerted by ambient pressure acting over the projected area of the shaft; i. e.,

$$C_L = \frac{W}{p_a L D} \tag{78}$$

Attitude Angle, ϕ

The attitude angle for a shaft rotating counter-clockwise, as shown in Fig. 1, is the angle between the load vector, $\vec{W}$, and the radius vector to the point of minimum film thickness; i. e.,

$$\phi = 2\pi - \theta_W \tag{79}$$

Friction Coefficient, C_f

The friction parameter is the ratio of the torque, M, actually required to rotate the shaft in its eccentric

position under load W, to the torque, M_o, which would be required to rotate the shaft if it were concentric with the bearing; i. e.,

$$C_f \equiv \frac{M}{M_o} \tag{80}$$

Eccentricity Ratio, ϵ

This parameter measures the lack of concentricity of the shaft with the bearing, and is the ratio of the difference between maximum and minimum film thicknesses ($h_{max} - h_{min}$) to the difference between bearing and shaft diameters (2c); i.e.,

$$\epsilon \equiv \frac{h_{max} - h_{min}}{2c} \tag{81}$$

Compressibility Bearing Parameter, Λ

This parameter is $3/\pi$ (practically unity) times the ratio of the torque for concentric operation, M_o, to the torque that would be produced by ambient pressure acting on the average cross-sectional film area, cL; i. e.,

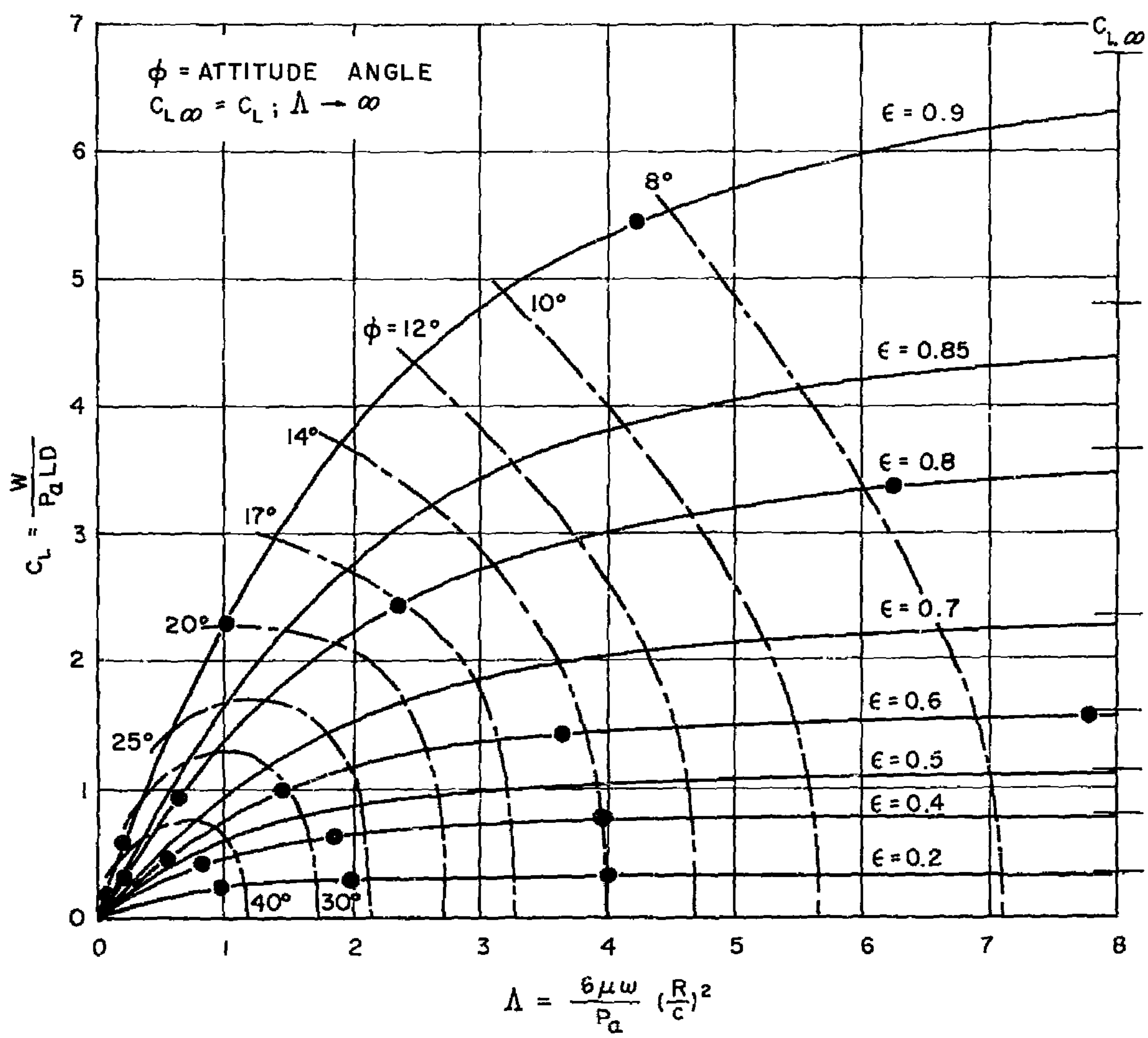
$$\Lambda \equiv \frac{3}{\pi} \frac{M_o}{p_a c L R} = \frac{6 \mu \omega}{p_a} \left(\frac{R}{c}\right)^2 \tag{82}$$

Table 6 summarizes the inter-relationships of the foregoing parameters, as found by digital computation for specific, discretely-spaced operating conditions. Design charts based on these data (suitably extrapolated by the formulas of Ausman and Katto-Soda.) are presented in Figs. 4, 5 and

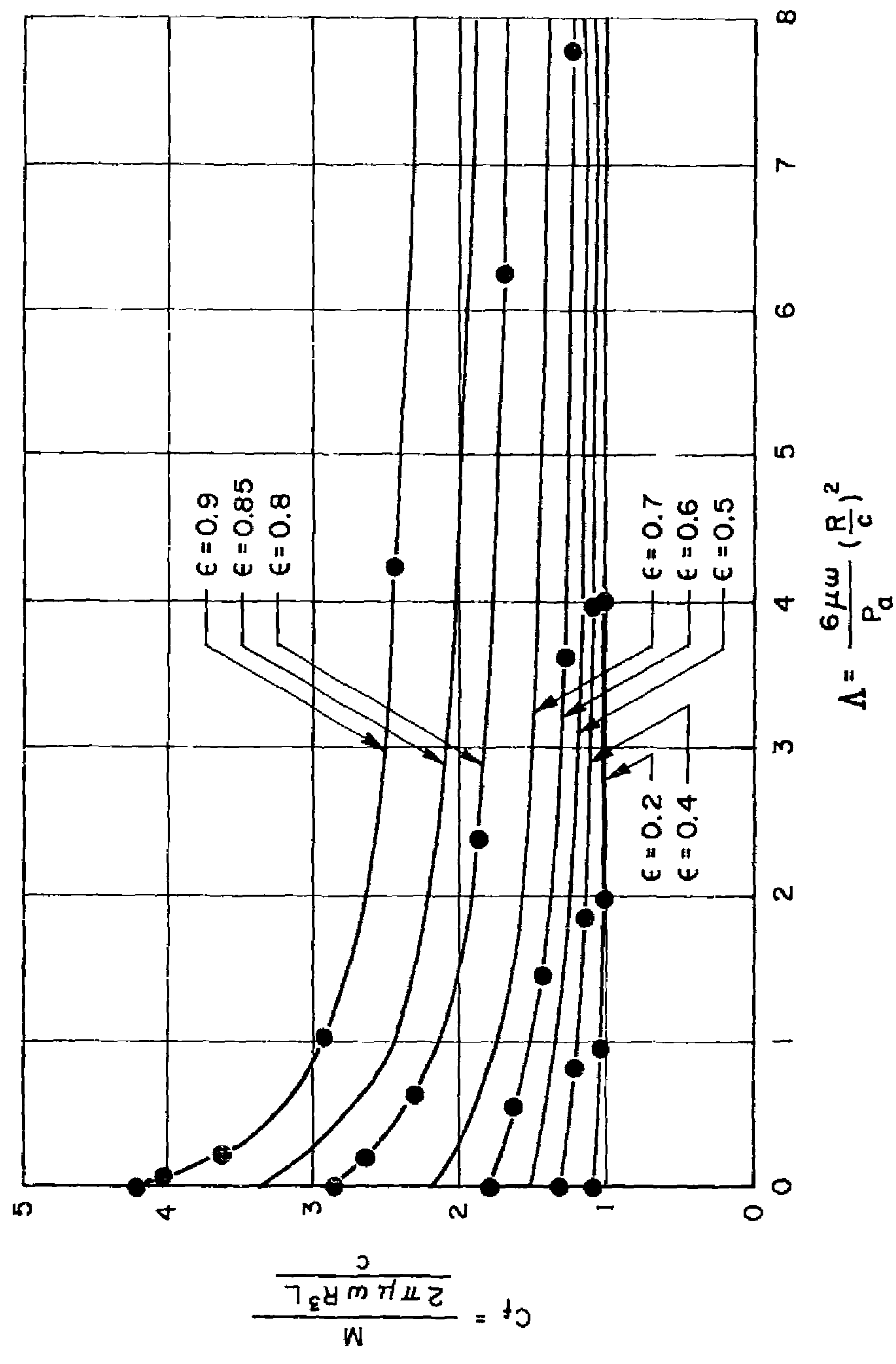
Table 6

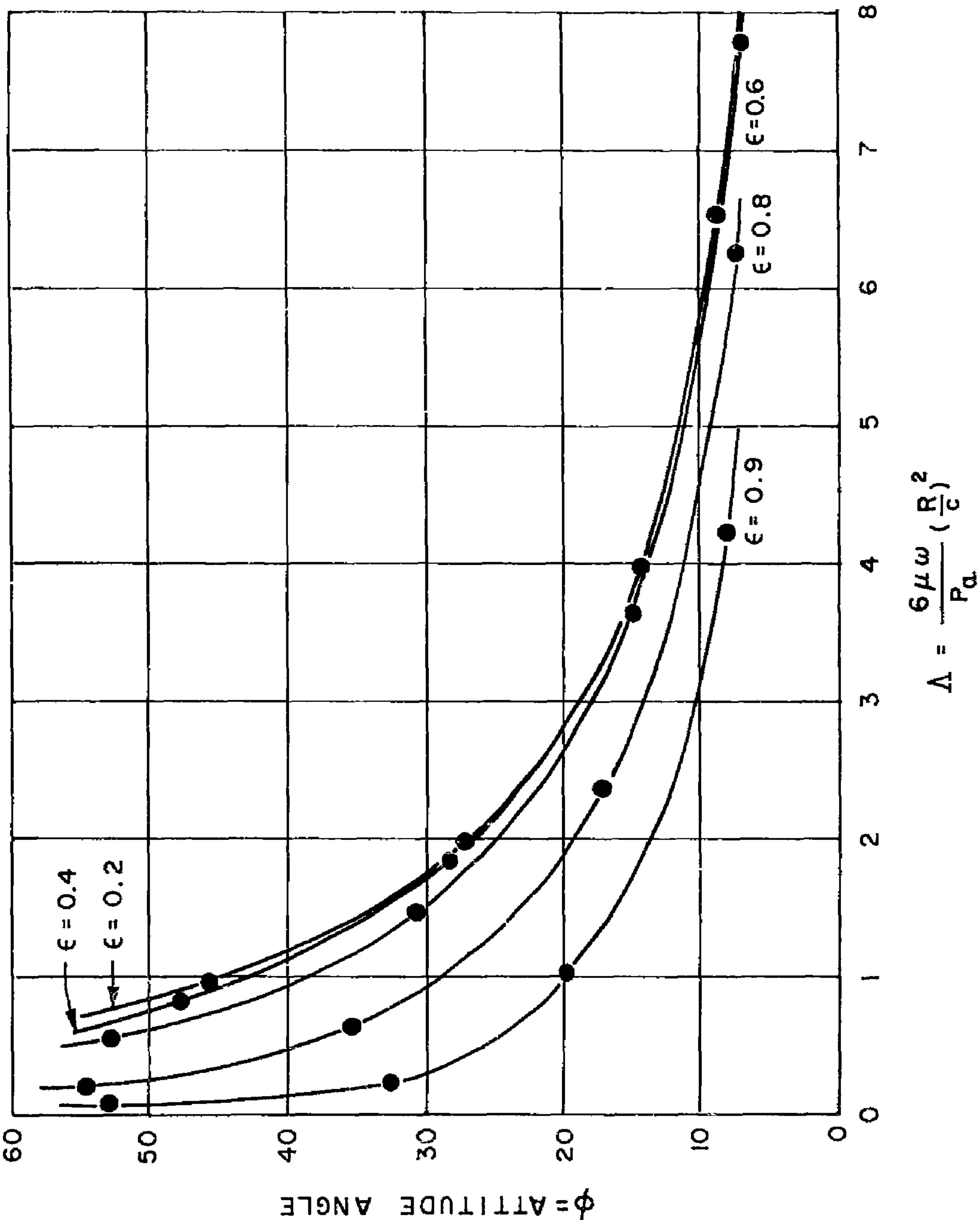
SUMMARY OF DIGITAL COMPUTER COMPUTATIONS

ν	ϵ	Λ	C_L	C_f	ϕ
1.0	0.2	0.9622	0.2207	1.0518	45.75
	0.4	0.8209	0.4156	1.2346	47.97
	0.6	0.5418	0.4518	1.6536	52.70
	0.8	0.2052	0.3039	2.6722	54.53
	0.9	0.0715	0.1708	4.0505	52.88
0.5	0.2	1.979	0.2914	1.0331	26.93
	0.4	1.859	0.6367	1.1528	28.10
	0.6	1.460	0.9967	1.4501	30.73
	0.8	0.6348	0.9537	2.3268	35.05
	0.9	0.2221	0.5933	3.6303	32.49
0.25	0.2	4.012	0.3214	1.0242	14.03
	0.4	3.976	0.7534	1.1090	14.38
	0.6	3.643	1.440	1.3090	15.10
	0.8	2.379	2.447	1.8987	17.23
	0.9	1.017	2.298	2.9412	19.80
0.125	0.2	8.058	0.3302	1.0214	7.13
	0.4	8.110	0.7869	1.0956	7.16
	0.6	7.781	1.581	1.2648	7.31
	0.8	6.256	3.379	1.7220	7.66
	0.9	4.246	5.461	2.4733	8.27



φ = ATTITUDE ANGLE
$C_{L\infty} = C_L ; \Lambda \to \infty$
$C_{L\infty}$
ε = 0.9
ε = 0.85
ε = 0.8
ε = 0.7
ε = 0.6
ε = 0.5
ε = 0.4
ε = 0.2
8°
10°
φ = 12°
14°
17°
20°
25°
40°
30°
$C_L = \frac{W}{P_a LD}$
$\Lambda = \frac{6\mu\omega}{P_a} (\frac{R}{c})^2$
0
1
2
3
4
5
6
7
8


ε = 0.9
ε = 0.85
ε = 0.8
ε = 0.7
ε = 0.6
ε = 0.5
ε = 0.2
ε = 0.4
$\Lambda = \frac{6\mu\omega}{P_a}\left(\frac{R}{c}\right)^2$
$C_f = \frac{M}{\frac{2\pi\mu\omega R^3 L}{c}}$


$\epsilon = 0.4$
$\epsilon = 0.2$
$\epsilon = 0.6$
$\epsilon = 0.8$
$\epsilon = 0.9$
ϕ = ATTITUDE ANGLE
$\Lambda = \frac{6 \mu \omega}{P_a} \left(\frac{R}{c}\right)^2$
0 1 2 3 4 5 6 7 8
0 10 20 30 40 50 60

6. With their use the designer can quickly arrive at an approximate bearing design. Approximate corrections for the differences in behavior between bearings of finite and of infinite length are given by Ausman[9] in a paper presented in this Symposium. Even when designed according to this method, the bearing may not perform satisfactorily because the predicted mode of steady-state operation is unstable. The theory of this "whirl" phenomenon is not yet fully understood, so that each bearing design should be "proven" by experiment. Further design considerations are given in ref. 9.

Several features of the curves in Fig. 4 are worthy of comment. First, at high values of the compressibility parameter, Λ, the load coefficient, C_L, becomes dependent on the eccentricity ratio, ϵ , only. (This fact might have been anticipated from the inequality in eq. 52.) Thus an increase of the rotational speed of a bearing may sometimes result in a negligible increase of load-carrying capacity. Also, if the speed, ambient pressure, overall dimensions and minimum film thickness of a bearing be maintained constant, an increase of load-carrying capacity may sometimes be achieved through an increase of clearance, c! Second, at small values of the eccentricity ratio the attitude angle becomes dependent on the compressibility parameter only. Thus the compressibility parameter sets the phase, so to speak, of the pressure distrib-

ution within the bearing relative to the film-thickness distribution in a manner somewhat similar to the way the free-stream Mach number affects the pressure distribution on a wave-shaped wall.[10] Third, at very low values of the compressibility parameter, the behavior of the complete, liquid-filled, non-cavitating journal bearing is approached. This region is crowded into the lower left-hand corner of Fig. 4.

With respect to Fig. 5, which depicts the characteristics of the friction coefficient, C_f, it should be observed that for small values of the eccentricity ratio the coefficient is nearly that for concentric operation. At $\Lambda = 0$, the "incompressible" coefficients are shown, as constructed from the data in ref. 11. The important features of the curves of attitude angle in Fig. 6 have already been discussed in connection with Fig. 4.

Acknowledgments:

The authors happily take this opportunity to express gratitude for the constant encouragement of Prof. D. D. Fuller, who introduced both to the field of gas bearings.

The extensive and very competent assistance of Messrs. S. Malanoski and C. Stevenson are hereby acknowledged. They prepared most of the tables and figures here presented.

The authors further wish to thank Messrs. L. McGinn and W. Ruegner of the Franklin Institute Univac staff for programming and computing the Katto-Soda solutions tabulated in this paper.

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APPENDIX I

Derivation of the Equations of Laminar Lubrication:

In this appendix the basic equations of laminar lubrication are derived from the over-all conservation equations of momentum, energy and mass for a single-phase Newtonian fluid. The geometry treated is that for the infinite slider bearing. Generalization of the derivation to include finite slider bearings is obvious, and generalization to include curved surfaces can no doubt be accomplished with general tensor techniques similar to those employed in ref. 7.

The physical situation to be analysed is shown in Fig. 7. The upper surface of the bearing is stationary.

Schematic Diagram of Slipper Bearing

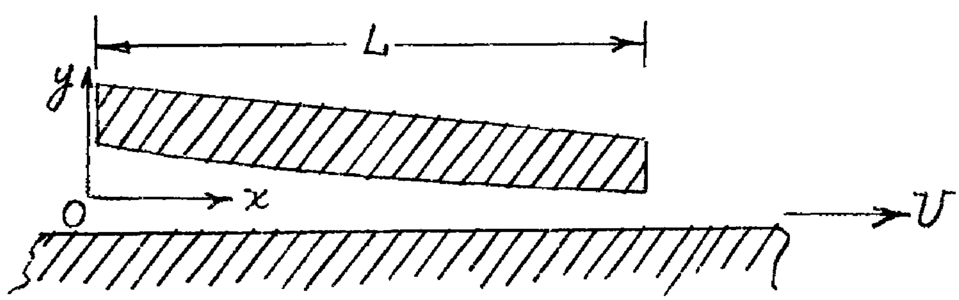

Figure 7

The lower surface moves in the x-direction with velocity U. The fluid pressures at the entrance and exit of the lubricating film have the same value, p*. The bearing surfaces are taken to be at the same uniform temperature, T*.

Basic Equations:

The equations governing the two-dimensional motion of

a Newtonian fluid are as follows (See, for example, ref. 5):

Conservation of Linear Momentum:

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}\right]+\frac{\partial}{\partial y}\left[\mu\left\{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right\}\right] \qquad (A1)$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}\right]+\frac{\partial}{\partial x}\left[\mu\left\{\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right\}\right] \qquad (A2)$$

Conservation of Mass:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \qquad (A3)$$

Conservation of Energy:

$$\rho C_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]+\left[\rho\left(\frac{\partial h}{\partial p}\right)_T-1\right]\left[u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right]=\frac{\partial}{\partial y}k\frac{\partial T}{\partial y}+\frac{\partial}{\partial x}k\frac{\partial T}{\partial x}+\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right] \qquad (A4)$$

The boundary conditions to be satisfied are as follows:

$$\begin{aligned} &u = U,\ v = 0,\ T = T^*;\ y = 0\\ &u = 0,\ v = 0,\ T = T^*;\ y = h(x) \qquad (A5)\\ &p = p^*;\ x = 0 \text{ or } L \end{aligned}$$

Formulation of Dimension-free Equations:

In order to determine the relative magnitudes of the terms in equations 1-4 in the limiting case of a very thin fluid film, it is convenient to convert all terms to appropriate dimensionless variables. First we define the new independent

variables: $\bar{x} = x/L$ and $\bar{y} = y/h(x)$ (A6)

Then, as in ref. 7, we hypothesize that the film thickness can be expressed in the form

$$h = h^* \exp\{g(\bar{x})\} \tag{A7}$$

where $g(\bar{x})$ and its derivatives are of O(1).

Now let ρ^* and ν^* be the fluid density and viscosity, respectively, corresponding to the state (p*,T*). Then define the new dependent variables:

$$\bar{u} = \frac{uL}{\nu^*} \; ; \quad \bar{v} = \frac{vL^2}{h^*\nu^*} \; ; \quad \bar{p} = \frac{p}{\rho^*\left(\frac{\nu^*}{h^*}\right)^2} \tag{A8}$$

Other barred quantities are to be made by dividing the local dimensional quantity by its value at the reference state (p*,T*). Thus: $\bar{k} = k/k(p^*,T^*)$, etc.

The barred variables are now substituted into eqs. 1-4. When these substitutions are made, the characteristic small parameter of fluid-dynamic lubrication appears; i. e.,

$$\delta = h^*/L \tag{A9}$$

The resulting equations are:

$$\bar{\rho}\left[\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{h^*}{h}\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}\right] = -\frac{1}{\delta^2}\left(\frac{\partial\bar{p}}{\partial\bar{x}}\right)_y + \frac{\partial}{\partial\bar{x}}\left[\bar{\mu}\left\{2\frac{\partial\bar{u}}{\partial\bar{x}} - \frac{2}{3}\left(\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)_y + \frac{h^*}{h}\frac{\partial\bar{v}}{\partial\bar{y}}\right)\right\}\right] + \frac{1}{\delta^2}\frac{\partial}{\partial\bar{y}}\bar{\mu}\frac{\partial\bar{u}}{\partial\bar{y}} + \frac{h^*}{h}\frac{\partial}{\partial\bar{y}}\bar{\mu}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)_y \tag{A10}$$

$$\bar{\rho}\left[\bar{u}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)_y + \frac{h^*}{h}\bar{v}\frac{\partial\bar{v}}{\partial\bar{x}}\right] = -\frac{1}{\delta^4}\frac{h^*}{h}\frac{\partial\bar{p}}{\partial\bar{y}} + \frac{1}{\delta^2}\frac{h^*}{h}\frac{\partial}{\partial\bar{y}}\left[\bar{\mu}\left\{2\frac{h^*}{h}\frac{\partial\bar{v}}{\partial\bar{y}} - \frac{2}{3}\left(\left(\frac{\partial\bar{u}}{\partial\bar{x}}\right)_y + \frac{h^*}{h}\frac{\partial\bar{v}}{\partial\bar{y}}\right)\right\}\right] + \frac{1}{\delta^2}\frac{\partial}{\partial\bar{x}}\bar{\mu}\frac{h^*}{h}\frac{\partial\bar{u}}{\partial\bar{y}} + \left(\frac{\partial}{\partial\bar{x}}\right)_y\bar{\mu}\left(\frac{\partial\bar{v}}{\partial\bar{x}}\right)_y \tag{A11}$$

$$\left.\frac{\partial}{\partial \bar{x}}\right)_y (\bar{\rho}\bar{u}) + \frac{h^*}{h}\frac{\partial}{\partial \bar{y}}(\bar{\rho}\bar{v}) = 0 \tag{A12}$$

$$\frac{c_p T^* L^2}{\nu^{*2}}\bar{\rho}\left[\bar{u}\left(\frac{\partial \bar{T}}{\partial \bar{x}}\right)_y + \frac{h^*}{h}\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right] + \left[\rho\left(\frac{\partial h}{\partial p}\right)_T - 1\right]\frac{1}{\delta^2}\left[\bar{u}\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right)_y + \frac{h^*}{h}\bar{v}\frac{\partial \bar{p}}{\partial \bar{y}}\right] =$$

$$\frac{k^* T^* L^2}{\mu^* \nu^{*2}}\left[\left.\frac{\partial}{\partial \bar{x}}\right)_y \bar{k}\left(\frac{\partial \bar{T}}{\partial \bar{x}}\right)_y + \frac{1}{\delta^2}\left(\frac{h^*}{h}\right)^2\frac{\partial}{\partial \bar{y}}\bar{k}\frac{\partial \bar{T}}{\partial \bar{y}}\right] + \bar{\mu}\left[2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_y^2 + 2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2\left(\frac{h^*}{h}\right)^2 +\right.$$

$$\left. + \left\{\delta\left(\frac{\partial \bar{v}}{\partial \bar{x}}\right)_y + \frac{1}{\delta}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{h^*}{h}\right\}^2 - \frac{2}{3}\left\{\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_y + \frac{h^*}{h}\frac{\partial \bar{v}}{\partial \bar{y}}\right\}^2\right] \tag{A13}$$

The boundary conditions to be satisfied are:

$$\begin{aligned} &\bar{u} = UL/\nu^*, \ \bar{v} = 0, \ \bar{T} = 1; \quad \bar{y} = 0 \\ &\bar{u} = 0 \qquad\quad , \ \bar{v} = 0, \ \bar{T} = 1; \quad \bar{y} = 1 \\ &\bar{p} = 1; \qquad\quad \bar{x} = 0 \text{ or } \bar{x} = 1 \end{aligned} \tag{A14}$$

<u>Expansion in δ</u> :

As in ref. 7, let us now restrict consideration to those solutions of the system 10-14 which can be expanded in "δ". Thus, we take:

$$\begin{aligned} u &= U_0 + \delta U_1 + \delta^2 U_2 \text{ etc.} \\ v &= V_0 + \delta V_1 + \ldots\ldots. \\ p &= P_0 + \delta P_1 + \ldots\ldots. \\ T &= T_0 + \delta T_1 + \ldots\ldots. \end{aligned} \tag{A15}$$

Here the capitalized functions are to be independent of "δ". The zeroth-order functions are made to assume all of the non-zero boundary conditions for their respective barred quantities so that all higher order functions must vanish on the boundaries.

Before the substitution of functions 15 into eqs. 10-14 is made, one should note that all $\bar{x}$ derivatives at constant "y" can be converted to $\bar{x}$ derivatives at constant "$\bar{y}$" without involvement of "δ". Thus if ψ is any function of position, we have:

$$\left(\frac{\partial \psi}{\partial \bar{x}}\right)_y = \left(\frac{\partial \bar{y}}{\partial \bar{x}}\right)_y \left(\frac{\partial \psi}{\partial \bar{y}}\right)_{\bar{x}} + \left(\frac{\partial \psi}{\partial \bar{x}}\right)_{\bar{y}} \quad \text{(A16)}$$

$$\left(\frac{\partial^2 \psi}{\partial \bar{x}^2}\right)_y = \left(\frac{\partial \bar{y}}{\partial \bar{x}}\right)_y \left(\frac{\partial^2 \psi}{\partial \bar{y}^2}\right)_{\bar{x}} + 2\left(\frac{\partial \bar{y}}{\partial \bar{x}}\right)_y \left(\frac{\partial^2 \psi}{\partial \bar{y}\,\partial \bar{x}}\right) + \left(\frac{\partial^2 \psi}{\partial \bar{x}^2}\right)_{\bar{y}} + \left(\frac{\partial^2 \bar{y}}{\partial \bar{x}^2}\right)_y \left(\frac{\partial \psi}{\partial \bar{y}}\right)_{\bar{x}}$$

(A17)

where: $$\left(\frac{\partial \bar{y}}{\partial \bar{x}}\right)_y = -\bar{y}\,\frac{\partial \ln h}{\partial \bar{x}} \quad \text{(A18)}$$

$$\left(\frac{\partial^2 \bar{y}}{\partial \bar{x}^2}\right)_y = \bar{y}\left(\frac{\partial \ln h}{\partial \bar{x}}\right)^2 - \bar{y}\left(\frac{\partial^2 \ln h}{\partial \bar{x}^2}\right) \quad \text{(A19)}$$

All functions of the thermodynamic state can be assumed to deviate from the state (P_0, T_0) by O(δ). Thus:

$$\bar{\rho} = \bar{\rho}(\bar{p}, \bar{T}) = \bar{\rho}_0(P_0, T_0) + \delta\left(\frac{\partial \bar{\rho}}{\partial \bar{p}}\right)_{\bar{T}} P_1 + \delta\left(\frac{\partial \bar{\rho}}{\partial \bar{T}}\right)_{\bar{p}} T_1 + \ldots \quad \text{(A20)}$$

When, now, the series 15 and 20 are substituted into eqs. 10-14, and the coefficients of the various powers of "δ" are equated to zero, we get:

$$\left(\frac{h^*}{h}\right)^2 \frac{\partial}{\partial \bar{y}} \bar{\mu}_0 \frac{\partial U_0}{\partial \bar{y}} = \left(\frac{\partial P_0}{\partial \bar{x}}\right)_y \quad \text{(A21)}$$

$$\frac{\partial P_0}{\partial \bar{y}} = 0 \quad \text{(A22)}$$

$$\left.\frac{\partial}{\partial \bar{x}}\right)_y (\rho_0 U_0) + \frac{h^*}{h} \frac{\partial}{\partial \bar{y}} (\rho_0 V_0) = 0 \quad \text{(A23)}$$

$$\left[\rho\left(\frac{\partial h}{\partial p}\right)_T - 1\right]\left[U_0\left(\frac{\partial P_0}{\partial \bar{x}}\right)_y + \frac{h^*}{h} V_0 \frac{\partial P_0}{\partial \bar{y}}\right] = \frac{k^* T^* L^2}{\mu^* \nu^*}\left(\frac{h^*}{h}\right)^2 \frac{\partial}{\partial \bar{y}} \bar{k}_0 \frac{\partial T_0}{\partial \bar{y}} + \bar{\mu}_0\left(\frac{h^*}{h}\right)^2\left(\frac{\partial U_0}{\partial \bar{y}}\right)^2 \quad \text{(A24)}$$

Similar, though more complicated, differential equations are obtainable for the higher order functions. The equations for laminar lubrication are obtained by approximating the barred quantities in eq. 15 by their zeroth order functions. When reversion is made to the original physical variables, and allowance is made for the additional space variable, "z," the resulting equations are eqs. 1-5 of the main text.

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